JACOB'S LADDERS AND THE ASYMPTOTIC FORMULA FOR THE INTEGRAL OF THE EIGHT ORDER EXPRESSION $|\zeta(1/2+i\varphi_2(t))|^4|\zeta(1/2+it)|^4$

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Abstract. It is proved in this paper that there is a fine correlation between the values of $|\zeta(1/2+i\varphi_2(t))|^4$ and $|\zeta(1/2+it)|^4$ where $\varphi_2(t)$ stands for the Jacob's ladder of the second order. This new asymptotic formula cannot be obtained in known theories of Balasubramanian, Heath-Brown and Ivic.

1. Results

Let $\mu(y)\in C^\infty([y_0,\infty))$ is a monotonically increasing (to $+\infty$) function, and let it obey $\mu(y)\geq 4y\ln y$. Similarly to [3], (3.1)-(3.9) we obtain that there exists an unique solution $x_\mu(T)=\varphi_2(T;\mu)=\varphi_2(T),\ T\geq T_0[\mu]$ to the nonlinear integral equation

$$\int_0^{\mu[x(T)]} Z^4(t)e^{-\frac{t}{x(T)}}\mathrm{d}t=\int_0^T Z^4(t)\mathrm{d}t. \tag{1.1}$$

Remark 1. The function $\varphi_2(T)$ is to be named the *Jacob's ladder of the second order*. This function obeys the following properties

(a) it is increasing for $T\geq T_0$,

(b) if $T=\gamma$ is a zero of the function $\zeta(1/2+iT)$ of the order $n(\gamma)$ then

$$\varphi_2'(\gamma)=\varphi_2''(\gamma)=\dots=\varphi_2^{(4n)}(\gamma)=0,\ \varphi_2^{(4n+1)}(\gamma)\neq 0,$$

(c) if

$$\Phi_2(y)=\int_0^{\mu(y)} Z^4(t)e^{-\frac{t}{y}}\mathrm{d}t,\ y=\varphi_2(T)$$

then

$$Z^4(t)=\Phi_2'[\varphi_2(T)]\frac{\mathrm{d}\varphi_2(T)}{\mathrm{d}T};\ \Phi_2'=\frac{\mathrm{d}\Phi_2}{\mathrm{d}\varphi_2},\ T\geq T_0, \tag{1.2}$$

where

$$\Phi_2'(y)=\frac{1}{y^2}\int_0^{\mu(y)} tZ^4(t)e^{-\frac{t}{y}}\mathrm{d}t+Z^4[\mu(y)]e^{-\frac{\mu(y)}{y}}\frac{\mathrm{d}\mu(y)}{\mathrm{d}y}. \tag{1.3}$$

The following theorem holds true

Theorem. If

$$[T,T+U]=\varphi_2\left\{\left[\mathring{T},\overset{\circ}{\widehat{T+U}}\right]\right\},\ U=T^{13/14+2\epsilon}, \tag{1.4}$$

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then

$$\int_{\mathring{T}}^{\widehat{\mathring{T+U}}}\left|\zeta\left(\frac{1}{2}+i\varphi_2(t)\right)\right|^4\left|\zeta\left(\frac{1}{2}+it\right)\right|^4\sim\frac{1}{4\pi^4}U\ln^8 T,\ T\to\infty. \tag{1.5}$$

Remark 2. The formula (1.5) is the first asymptotic formula in the theory of the Riemann zeta-function for the eight order expression

$$\left|\zeta\left(\frac{1}{2}+i\varphi_2(t)\right)\right|^4\left|\zeta\left(\frac{1}{2}+it\right)\right|^4=Z^4[\varphi_2(t)]Z^4(t),$$

where

$$Z(t)=e^{i\vartheta(t)}\zeta\left(\frac{1}{2}+it\right),\ \vartheta(t)=-\frac{t}{2}\ln\pi+\operatorname{Im}\ln\Gamma\left(\frac{1}{4}+i\frac{t}{2}\right).$$

It is clear that the formula (1.5) cannot be obtained by complicated methods of Balasubramanian, Heath-Brown and Ivic (see [2]).

This paper is a continuation of the series of works [3]-[9].

2. Consequences of the Titchmarsh-Atkinson formula

Titchmarsh has proved in 1928 the following formula (see [10], pp. 137, 141, [11], p. 143)

$$\int_0^\infty Z^4(t)e^{-\delta t}\mathrm{d}t\sim\frac{1}{2\pi^2}\frac{1}{\delta}\ln^4\frac{1}{\delta}. \tag{2.1}$$

Let us remind the Titchmarsh-Atkinson formula (see [11], p. 145)

$$\int_0^\infty Z^4(t)e^{-\delta t}\mathrm{d}t=\frac{1}{\delta}\left(A\ln^4\frac{1}{\delta}+B\ln^3\frac{1}{\delta}+C\ln^2\frac{1}{\delta}+D\ln\frac{1}{\delta}+E\right)+ \mathcal{O}\left\{\left(\frac{1}{\delta}\right)^{13/14+\epsilon}\right\},\ A=\frac{1}{2\pi^2} \tag{2.2}$$

which improved the Titchmarsh formula (2.1). The following lemma is true.

Lemma 1.

$$\int_0^T Z^4(t)\mathrm{d}t=\varphi_2(T)\left\{A\ln^4\varphi_2(T)+B\ln^3\varphi_2(T)+C\ln^2\varphi_2(T)+ +D\ln\varphi_2(T)+E\right\}+\mathcal{O}\left\{(\varphi_2(T))^{13/14+\epsilon}\right\},\ A=\frac{1}{2\pi^2}. \tag{2.3}$$

Proof. Similarly to [3], (4.3)-(4.6) we have

$$Z^4(t)e^{-\frac{\delta}{2}t}<te^{-\frac{\delta}{2}t}=f_2(t;\delta)\le f_2\left(\frac{2}{\delta};\delta\right)=\frac{2}{e\delta},$$

$$\int_U^\infty Z^4(t)e^{-\frac{\delta}{2}t}e^{-\frac{\delta}{2}t}\mathrm{d}t<\frac{4}{e\delta^2}e^{-\frac{\delta}{2}U}.$$

The value $U=\mu(1/\delta)$ is to be chosen by the following rule

$$\frac{4}{e\delta^2}e^{-\frac{\delta}{2}U}\le\frac{1}{\sqrt{\delta}}\ \Rightarrow\ \mu\left(\frac{1}{\delta}\right)\ge\frac{4}{\delta}\ln\frac{1}{\delta}>\frac{2}{\delta}\ln\frac{4}{e\delta^{3/2}}.$$

Now (2.2) implies

$$\int_0^{\mu(1/\delta)} Z^4(t)e^{-\delta t}\mathrm{d}t = \frac{1}{\delta}\left(A\ln^4\frac{1}{\delta} + B\ln^3\frac{1}{\delta} + C\ln^2\frac{1}{\delta} + \right. \tag{2.4}$$

$$\left. + D\ln\frac{1}{\delta} + E\right) + \mathcal{O}\left\{\left(\frac{1}{\delta}\right)^{13/14+\epsilon}\right\},\ \mu\left(\frac{1}{\delta}\right) \geq \frac{4}{\delta}\ln\frac{1}{\delta}.$$

Since (see (1.1), compare [3], (3.3))

$$\int_0^{\mu(1/\delta)} Z^4(t)e^{-\delta t}\mathrm{d}t = \int_0^{M_2(y)} Z^4(t)\mathrm{d}t,$$

and $\frac{1}{\delta} = y = \varphi_2(T),\ M_2[\varphi_2(T)] = T$, then (2.4) implies (2.3). □

3. The asymptotic formula $\varphi_2(T) \sim T$

The following lemma is true.

Lemma 2.

$$\varphi_2(T) - T = \mathcal{O}\left(\frac{T}{\ln T}\right),\ T \to \infty. \tag{3.1}$$

Proof. In 1924 Ingham has proved the following formula (see [1], p. 277, [11], p. 125)

$$\int_0^T Z^4(t)\mathrm{d}t = \frac{1}{2\pi^2}T\ln^4 T + \mathcal{O}(T\ln^3 T). \tag{3.2}$$

Let us remind the Ingham-Heath-Brown formula (see [2], p. 129)

$$\int_0^T Z^4(t)\mathrm{d}t = T\sum_{k=0}^4 C_k \ln^{4-k} T + \mathcal{O}(T^{7/8+\epsilon}), \tag{3.3}$$

which improved the Ingham formula (3.2); $C_0 = \frac{1}{2\pi^2}$ is the Ingham constant. Putting $T = M_2(y); \varphi_2(T) = \varphi_2[M_2(y)] = y$ into eq. (2.3) we obtain

$$\int_0^{M_2(y)} Z^4(t)\mathrm{d}t = y\left(C_0\ln^4 y + B\ln^3 y + C\ln^2 y + D\ln y + E\right) + \mathcal{O}(y^{13/14+\epsilon}). \tag{3.4}$$

Furthermore, putting $T = \frac{4}{5}y, \frac{5}{4}y$ into eq. (3.3) and comparing with the formula (3.4) we obtain

$$\frac{4}{5}y < M_2(y) < \frac{5}{4}y \ \Rightarrow\ \frac{4}{5}\varphi_2(T) < T < \frac{5}{4}\varphi_2(T) \ \Rightarrow\ \frac{4}{5}T < \varphi_2(T) < \frac{5}{4}T,$$

i.e.

$$|\varphi_2(T) - T| \leq \frac{1}{4}T. \tag{3.5}$$

Now, by comparing of the formulae (2.3), (3.3) (see (3.5)) we obtain

$$C_0\left\{\varphi_2(T)\ln^4\varphi_2(T) - T\ln^4 T\right\} = \mathcal{O}(T\ln^3 T). \tag{3.6}$$

Next, from (3.6) by the Taylor formula and (3.5), we obtain

$$C_0\left(\ln^4 T + 4\ln^3 T\right)[\varphi_2(T) - T] + \mathcal{O}\left\{\frac{\ln^3\hat{T}}{\hat{T}}[\varphi_2(T) - T]^2\right\} = \tag{3.7}$$

$$\mathcal{O}(T\ln^3 T),\ \hat{T} = \mathcal{O}(T).$$

Finally, we obtain (3.1) from (3.7). □

4. LEMMA ABOUT $\Phi''_{y^2}[\varphi_2(T)]$

By (1.3) we have

$$\Phi''_{y^2}(y) = J + Q, \tag{4.1}$$

where

$$J = \frac{1}{y^3}\int_0^{\mu(y)} \left(\frac{t^2}{y} - 2t\right) e^{-\frac{t}{y}} Z^4(t)\mathrm{d}t, \tag{4.2}$$

$$\begin{aligned} Q = e^{-\frac{\mu(y)}{y}} \left\{ \frac{2}{y^2}\mu(y)\frac{\mathrm{d}\mu(y)}{\mathrm{d}y} Z^4[\mu(y)] + 4\left(\frac{\mathrm{d}\mu(y)}{\mathrm{d}y}\right)^2 Z^3[\mu(y)]Z'[\mu(y)] - \right. \\ \left. -\frac{1}{y}\left(\frac{\mathrm{d}\mu(y)}{\mathrm{d}y}\right)^2 Z^4[\mu(y)] + \frac{\mathrm{d}^2\mu(y)}{\mathrm{d}y^2} Z^4[\mu(y)] \right\}. \end{aligned} \tag{4.3}$$

The following lemma is true.

Lemma 3. *If $\mu(y) = 4y\ln y$ then*

$$\Phi''_{y^2}[\varphi_2(T)] = \mathcal{O}\left\{\frac{1}{T}\ln^4 T(\ln\ln T)^2\right\}. \tag{4.4}$$

Proof. Let

$$g_1(t) = \left(\frac{t^2}{y} - 2t\right) e^{-\frac{t}{y}},\ t \in [0, \mu(y)].$$

We apply the following elementary facts

$$\begin{aligned} &\min\{g_1(t)\} = g_1[(2-\sqrt{2})y] = -2(\sqrt{2}-1)e^{-2+\sqrt{2}}y, \\ &\max\{g_1(t)\} = g_1[(2+\sqrt{2})y] = 2(\sqrt{2}+1)e^{-2-\sqrt{2}}y, \\ &g_1(t) \le g_1(y\ln\ln y) < y\frac{(\ln\ln y)^2}{\ln y},\ t \in [y\ln\ln y, 4y\ln y], \end{aligned} \tag{4.5}$$

and the Ingham formula (see (3.1)). First of all we have (see (4.2))

$$\begin{aligned} \frac{1}{y^3}\int_0^{y\ln\ln y} &= \mathcal{O}\left(\frac{1}{y^2}\int_0^{y\ln\ln y} Z^4(t)\mathrm{d}t\right) = \mathcal{O}\left(\frac{1}{y^2}y\ln^4 y\ln\ln y\right) = \\ &= \mathcal{O}\left(\frac{1}{y}\ln^4 y\ln\ln y\right), \end{aligned} \tag{4.6}$$

$$\begin{aligned} \frac{1}{y^3}\int_{y\ln\ln y}^{4y\ln y} &= \mathcal{O}\left(\frac{1}{y^3}y\frac{(\ln\ln y)^2}{\ln y}\int_0^{4y\ln y} Z^4(t)\mathrm{d}t\right) = \\ &= \mathcal{O}\left(\frac{1}{y^2}\frac{(\ln\ln y)^2}{\ln y}y\ln^5 y\right) = \mathcal{O}\left(\frac{1}{y}\ln^4 y(\ln\ln y)^2\right) \end{aligned} \tag{4.7}$$

by (3.1), (4.5). Next we have (see (4.3))

$$Q(y) = \mathcal{O}\left(\frac{\ln^3 y}{y^3}\right). \tag{4.8}$$

Finally, we obtain (4.4) from (4.1) by (4.2), (4.3), (4.6)-(4.8). □

Remark 3. It is quite evident that our lemma (i.e. also Theorem) is true for continuous class of functions

$$\mu(y) = 4y^{\omega_1}\ln^{\omega_2} y,\ \omega_1,\omega_2 \geq 1.$$

5. Proof of the Theorem

5.1. Since

$$G(t) = t(A\ln^4 t + B\ln^3 t + C\ln^2 t + D\ln t + E) \ \Rightarrow$$
$$G'(t) = A\ln^4 t + \mathcal{O}(\ln^3 t),$$

then from (2.3) we obtain (see (3.1))

$$\begin{aligned}(5.1)\quad \int_T^{T+U} Z^4(t)\mathrm{d}t &= \left[A\ln^4\xi + \mathcal{O}(\ln^3\xi)\right][\varphi_2(T+U)-\varphi_2(T)] + \mathcal{O}(T^{13/14+\epsilon}) \\ &= \left[C_0\ln^4\xi + \mathcal{O}(\ln^3\xi)\right]U\tan[\alpha_2(T,U)] + \mathcal{O}(T^{13/14+\epsilon}),\end{aligned}$$

where

$$(5.2)\qquad U = T^{13/14+2\epsilon},\ A = C_0 = \frac{1}{2\pi^2},\ \xi \in (\varphi_2(T),\varphi_2(T+U)),$$
$$\tan[\alpha_2(T,U)] = \frac{\varphi_2(T+U)-\varphi_2(T)}{U},$$

and $\alpha_2(T,U)$ is the angle of the chord of the curve $y=\varphi_2(T)$ that binds the points $[T,\varphi_2(T)]$ and $[T+U,\varphi_2(T+U)]$.
Next we have (see (3.3))

$$(5.3)\qquad \int_T^{T+U} Z^4(t)\mathrm{d}t = C_0 U\ln^4 T + \mathcal{O}(U\ln^3 T).$$

Comparing formulae (5.1), (5.3) we obtain

$$(5.4)\qquad \tan[\alpha_2(T,U)] = \frac{\ln^4\xi + \mathcal{O}(\ln^3\xi)}{\ln^4 T + \mathcal{O}(\ln^3 T)} + \mathcal{O}\left(\frac{1}{T^\epsilon \ln^4 T}\right).$$

Next we have (see (3.1), (5.2))

$$\ln\xi = \ln\{\varphi_2(T) + \xi - \varphi_2(T)\} = \ln\varphi_2(T) + \mathcal{O}\left\{\frac{\varphi_2(T+U)-\varphi_2(T)}{\varphi_2(T)}\right\} =$$
$$= \ln T + \mathcal{O}\left(\frac{1}{\ln T}\right),$$

i.e.

$$(5.5)\qquad \ln^4\xi = \ln^4 T + \mathcal{O}(\ln^3 T);\ \ln^3\xi = \mathcal{O}(\ln^3 T).$$

Hence we have

$$(5.6)\qquad \tan[\alpha_2(T,U)] = 1 + \mathcal{O}\left(\frac{1}{\ln T}\right).$$

5.2. Next we have (see (1.2))

$$\int_T^{T+U} Z^4(t)\mathrm{d}t = \int_T^{T+U} \Phi_2'[\varphi_2(t)]\mathrm{d}\varphi_2(t) =$$
$$\Phi_2'[\varphi_2(\eta)][\varphi_2(T+U)-\varphi_2(T)] = \Phi_2'[\varphi_2(\eta)]U\tan[\alpha_2(T,U)],\ \eta\in(T,T+U),$$

i.e. (see (5.6))

$$\int_T^{T+U} Z^4(t)\mathrm{d}t = \Phi_2'[\varphi_2(\eta)]\left\{1+\mathcal{O}\left(\frac{1}{\ln T}\right)\right\}U,\ U=T^{13/14+2\epsilon}. \tag{5.7}$$

Comparing formulae (5.3) and (5.7) we obtain

$$\Phi_2'[\varphi_2(\eta)] = C_0\ln^4 T + \mathcal{O}(\ln^3 T). \tag{5.8}$$

We have next (see (4.4))

$$\begin{aligned} &\Phi_2'[\varphi_2(t)] - \Phi_2'[\varphi_2(\eta)] = \Phi_2''[\varphi_2(\rho)][\varphi_2(t)-\varphi_2(\eta)] = \\ &\mathcal{O}\left\{\frac{1}{T}\ln^4 T(\ln\ln T)^2|\varphi_2(t)-\varphi_2(\eta)|\right\},\ t,\eta,\rho\in[T,T+U]. \end{aligned} \tag{5.9}$$

Since (see (3.1))

$$\varphi_2(t)-\varphi_2(\eta) = t-\eta+\mathcal{O}\left(\frac{T}{\ln T}\right) = \mathcal{O}\left(\frac{T}{\ln T}\right)$$

then we obtain from (5.9) by (5.8)

$$\begin{aligned} &\Phi'[\varphi_2(t)] = \Phi_2'[\varphi_2(\eta)] + \mathcal{O}\left\{\ln^3 T(\ln\ln T)^2\right\} = \\ &C_0\ln^4 T + \mathcal{O}\left\{\ln^3 T(\ln\ln T)^2\right\}. \end{aligned} \tag{5.10}$$

Hence, we obtain from the formula (see (1.2))

$$Z^4(t) = \Phi_2'[\varphi_2(t)]\frac{\mathrm{d}\varphi_2(t)}{\mathrm{d}t},\ t\in[T,T+U]$$

by (5.10) the main formula

(5.11)

$$Z^4(t) = C_0\ln^4 T\left\{1+\mathcal{O}\left(\frac{(\ln\ln T)^2}{\ln T}\right)\right\}\frac{\mathrm{d}\varphi_2(t)}{\mathrm{d}t},\ t\in[T,T+U],\ U=T^{13/14+2\epsilon}.$$

5.3. Putting

$$\mathcal{Z}^4(t) = \frac{Z^4(t)}{1+\mathcal{O}\left\{\frac{(\ln\ln T)^2}{\ln T}\right\}}, \tag{5.12}$$

we obtain (see (5.11))

$$\mathcal{Z}^4(t) = C_0\ln^4 T\frac{\mathrm{d}\varphi_2(t)}{\mathrm{d}t},\ t\in[T,T+U]. \tag{5.13}$$

Then from (5.13) by (5.3) we obtain the following $\mathcal{Z}^4$-transformation

$$\begin{aligned} &\int_{\mathring{T}}^{\widehat{\overset{\circ}{T+U}}} Z^4[\varphi_2(t)]\mathcal{Z}^4(t)\mathrm{d}t = C_0\ln^4 T\int_{\mathring{T}}^{\widehat{\overset{\circ}{T+U}}} Z^4[\varphi_2(t)]\frac{\mathrm{d}\varphi_2(t)}{\mathrm{d}t}\mathrm{d}t = \\ &C_0\ln^4 T\int_T^{T+U} Z^4(t)\mathrm{d}t = C_0^2U\ln^8 T + \mathcal{O}(U\ln^7 T). \end{aligned} \tag{5.14}$$

Since (see (5.12))

$$\int_{\mathring{T}}^{\overset{\circ}{\widehat{T+U}}} Z^4[\varphi_2(t)]\mathcal{Z}^4(t)\mathrm{d}t = \tag{5.15}$$

$$= \frac{1}{1+\mathcal{O}\left\{\frac{(\ln\ln\tau)^2}{\ln\tau}\right\}} \int_{\mathring{T}}^{\overset{\circ}{\widehat{T+U}}} Z^4[\varphi_2(t)]Z^4(t)\mathrm{d}t,\ \tau\in(\mathring{T},\overset{\circ}{\widehat{T+U}}), \tag{5.16}$$

and $T\to\infty \ \Rightarrow\ \tau\to\infty;\ \varphi_2(\mathring{T})=T$, then the formula (1.5) follows from (5.14) by (5.15).

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REFERENCES

[1] A.E. Ingham, 'Mean-value theorems in the theory of the Riemann zeta-function', Proc. Lond. Math. Soc. 2, 27 (1926) 273-300.

[2] A. Ivic, 'The Riemann zeta-function', A Willey-Interscience Publication, New York, 1985.

[3] J. Moser, 'Jacob's ladders and the almost exact asymptotic representation of the Hardy-Littlewood integral', (2008), arXiv:0901.3973.

[4] J. Moser, 'Jacob's ladders and the tangent law for short parts of the Hardy-Littlewood integral', (2009), arXiv:0906.0659.

[5] J. Moser, 'Jacob's ladders and the multiplicative asymptotic formula for short and microscopic parts of the Hardy-Littlewood integral', (2009), arXiv:0907.0301.

[6] J. Moser, 'Jacob's ladders and the quantization of the Hardy-Littlewood integral', (2009), arXiv:0909.3928.

[7] J. Moser, 'Jacob's ladders and the first asymptotic formula for the expression of the sixth order $|\zeta(1/2+i\varphi(t)/2)|^4|\zeta(1/2+it)|^2$', (2009), arXiv:0911.1246.

[8] J. Moser, 'Jacob's ladders and the first asymptotic formula for the expression of the fifth order $Z[\varphi(t)/2+\rho_1]Z[\varphi(t)/2+\rho_2]Z[\varphi(t)/2+\rho_3]\hat{Z}^2(t)$ for the collection of disconnected sets', (2009), arXiv:0912.0130.

[9] J. Moser, 'Jacob's ladders, the iterations of Jacob's ladder $\varphi_1^k(t)$ and asymptotic formulae for the integrals of the products $Z^2[\varphi_1^n(t)]Z^2[\varphi^{n-1}(t)]\cdots Z^2[\varphi_1^0(t)]$ for arbitrary fixed $n\in\mathbb{N}$' (2010), arXiv:1001.1632.

[10] E.C. Titchmarsh, 'The mean-value of the zeta-function on the critical line'. Proc. Lond. Math. Soc. 2, 27 (1928), 137-150.

[11] E.C. Titchmarsh, 'The theory of the Riemann zeta-function', Clarendon Press, Oxford, 1951.

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